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Two-Dimensional Ising Models with Layered Quenched Bond Randomness I.

—The Devil's Staircase of the Distribution Function and the Specific Heat for Random Ferromagnets—

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Abstract

Exactly solvable two-dimensional Ising models with McCoy-Wu type layered quenched bond randomness are studied both analytically and numerically. For binary type bond distributions, it is proved that the distribution function of a random variable, which appears in the exact solution, shows the devil's staircase structure for certain range of temperatures. By using a direct iteration method, the free energy and the specific heat are calculated in such a way as to give the numerically exact solution for a finite system. The specific heat does not show divergent behavior around the 'critical' temperature. A explanation in terms of the behavior of the distribution function is given. The boundary magnetization is also obtained exactly.

§ 1. Introduction

In the past decade, considerable amount of studies, both theoretically and experimentally, are made on the random spin systems. Among other things, the spin glasses have attracted much attention. Particularly, a novel type of ordered phase, the spin glass phase has been expected to appear. Although numerous theoretical works have been done, decisive answers to the fundamental problems, e.g., the nature of the phase transition and even the existence of the spin glass order itself, have not been given yet. Main difficulty which makes reliable theoretical analyses inaccessible lies in handling averages with respect to quenched randomness.

In such a situation, exactly solvable models, although unrealistic in some sence, are of precious value to elucidate certain aspects of the problem. An important example is the Sherrington-Kirkpatrick (SK) model[1] for the spin glasses whose exact solution has revealed, through recent studies, many of its interesting features[2].

Since the SK model is an infinite-ranged one and then would be regarded as a model in the infinite space dimensions, its behavior might be somewhat different from short-ranged systems in realistic (three or two) dimensions. Hence tractable models with short-ranged interactions are required. In this paper, we take a two-dimensional model, McCoy-Wu type random Ising model[3),4)], which meet this requirement.

McCoy and Wu presented the model and analyzed it for the case in which the bond distribution function has a sharp single peak and showed that the critical behavior changes due to the effect of randomness. Namely, they found an essential singularity in the specific heat at the critical temperature. For more realistic randomness, e.g. binary type randomness, no attempts have been made to study the thermodynamical behavior of the system. Although the exact 'critical' temperature T_c has been located for the original model and its generalizations[5)], the physical meaning of the T_c has not been clarified yet. We also note that the original model by McCoy-Wu allows extensions so as to include frustrations, and then the model can be applied to the spin glass problem. Hence it is important to study the thermodynamical behavior of the generalized models as well as the original model.

The paper is organized as follows. In § 2, a brief explanation of the models we consider is given. Necessary known results of the models are summarized. In § 3, the solution of the 'integral' equation for a distribution function is given. A 'phase transition' in the behavior of the distribution function was found. In § 4, thermodynamic quantities are calculated by a direct iteration method. The 'critical behavior' of the specific heat is discussed. The last section is devoted to summary and discussion. In the appendix we discuss the boundary magnetization briefly.

§ 2. McCoy-Wu type two-dimensional random Ising models

2.1 Models and their classification

We consider a two-dimensional square-lattice Ising spin system with the Hamiltoniam

$$H = -\sum_{i,j} J^V(j)\sigma_{i,j}\cdot\sigma_{i,j+1} - \sum_{i,j} J^H(j)\sigma_{i,j}\cdot\sigma_{i+1,j} \tag{1}$$

where $\sigma_{i,j}$ $(=\pm 1)$ is the spin at the site (i,j) and $J^V(j)$ ($J^H(j)$) is the vertical (horizontal) bond strength which is allowed to vary row to row. The system size is N (horizontal) ×M (vertical) and the boundary condition is chosen to be cyclic (free) in the horizontal (vertical) direction. We regard both $\{J^V(j)\}$ and $\{J^H(j)\}$ as sets of independent random variables. We consider the following three types of randomness:

(a) J^V is random and J^H is non-random: type V (vertical).

(b) J^H is random and J^V is non-random: type H (horizontal).

(c) Both J^V and J^H are random: type VH.

The original model of McCoy-Wu belongs to type V. If we allow J^H to take negative values models in type H or VH contain frustrations. However, in this paper, we restrict ourselves to the cases where both J^V and J^H take only positive values (i.e. the case of random ferromagnets). We denote the inverse temperature $1/(k_B T)$ by β and set $k_B=1$ throughout this paper.

The free energy per site F can be calculated by the use of the Pfaffian method[3] as

$$\begin{aligned} F = & \lim_{M\to\infty}\left(-\frac{1}{\beta}\right)\frac{1}{M}\sum_j^M \ln\{2\cosh\beta J^V(j)\} + \lim_{M\to\infty}\left(-\frac{1}{\beta}\right)\frac{1}{M}\sum_j^M \ln\{\cosh\beta J^H(j)\} \\ & + \lim_{M\to\infty}\left(-\frac{1}{\beta}\right)\frac{1}{4\pi M}\sum_j^{M-1}\int_{-\pi}^{\pi} d\theta \ln(1+w_j^2+2w_j\cos\theta) \\ & + F_{\text{sing}} \ , \end{aligned} \tag{2a}$$

$$F_{\text{sing}} = \lim_{M\to\infty}\left(-\frac{1}{\beta}\right)\frac{1}{4\pi M}\sum_j^{M-1}\int_{-\pi}^{\pi} d\theta \ln(a_j^2+b_j^2+a_j\lambda_j y_j) \ , \tag{2b}$$

where

$$w_j = \tanh(\beta J^H(j)), \tag{3a}$$
$$\lambda_j = \tanh^2(\beta J^V(j)), \tag{3b}$$
$$a_j(\theta) = -2w_j \sin\theta |1+w_j e^{i\theta}|^{-2}, \tag{3c}$$
$$b_j(\theta) = (1-w_j^2)|1+w_j e^{i\theta}|^{-2}, \tag{3d}$$

and the random variables $\{y_j\}$ are defined iteratively as

$$y_{j+1} = F[y_j | J^V(j), J^H(j)] \ , \tag{4a}$$
$$F[x|J^V(j), J^H(j)]) = \frac{a_j + \lambda_j x}{a_j^2 + b_j^2 + a_j \lambda_j x} \ , \tag{4b}$$

with $y_0 = 0$. The above explicit expression of the free energy has not been given before, although implicit in some literatures[4,5]. Furstenberg's theorem[6] assures that the distribution function ν_j of the variable y_j defined by

$$\nu_j(y) = \langle\langle \delta(y-y_j) \rangle\rangle, \tag{5}$$

where $\langle\langle \cdots \rangle\rangle$ denotes the average with respect to $\{J^V(j)\}$ and $\{J^H(j)\}$, has its limit $\nu(y)$ as $j \to \infty$:

$$\lim_{j\to\infty} \nu_j(y) = \nu(y) \ . \tag{6}$$

Various thermodynamic quantities, such as the specific heat, the boundary magnetization and the boundary correlation function, are expressed[3] by the integrals involving the function $\nu(y)$. Hence the detailed study of the function $\nu(y)$ is important to know the thermodynamical behavior of the model. Similar kinds of distribution functions were investigated in the study of general 2x2 random matrices[7], but actual calculations for McCoy-Wu type random Ising model has not been done.

2.2 The devil's staircase structure of the distribution function

In the rest of this section, we restrict ourselves to V-and H-type models with binary bond distributions. Since the argument is almost the same for both types, we take the case of V-type in the following.

The bond distribution function $P(J^V)$ is taken to be as

$$P(J^V) = (1-c)\delta(J^V - J_1^V) + c\delta(J^V - J_2^V), \tag{7}$$

with $0<c<1$ and set $J^H(j) = J^H$, $w_j = w$, $a_j(\theta) = a(\theta)$ and $b_j(\theta) = b(\theta)$ for all j.

First we discuss the behavior of the limiting distribution of y. For this purpose, it is convenient to work on the integrated distribution function N(y) defined by

$$N(y) = \int_{-\infty}^{y} \nu(y')dy'. \tag{8}$$

From Eq.(4), we see that N(y) satisfies the equation

$$N(y)=(1-c)N[f_1(y)]+cN[f_2(y)], \tag{9}$$

where $f_1(x)$ ($f_2(x)$) is the inverse function of $F[x\mid J_1^V, J^H]$ ($F[x\mid J_2^V, J^H]$). The Eq.(9) is of the same type as the one discussed by Bruinsma and Aeppli[8] for a one-dimensional random field Ising model. The solution has the devil's staircase (DS)structure for certain range of the parameter values. Due to the symmetry $\nu(y;\theta)=\nu(-y;-\theta)$, we can restrict the range of θ to $-\pi<\theta<0$, and we take $0<J_1^V<J_2^V$ without loss of generality. Under these conditions we find that both $f_1(x)$ and $f_2(x)$ are monotonically increasing functions with $f_1(x)>f_2(x)$ for $x>0$. Thus, the argument of Bruinsma and Aeppli can be directly applied and we present the results briefly.

The solution of Eq.(9) is given iteratively. Let us denote the lower (upper) limit of the p-th plateau in the m-th iteration level by $y_L(m,p)$ ($y_U(m,p)$) and its height (or amplitude) by $A(m,p)$. We have ($m\geqq 2$),

for $1\leqq p\leqq 2^{m-2}$

$$y_L(m,p)=f_1^{-1}[y_L(m-1,p)], \tag{10a}$$
$$y_U(m,p)=f_1^{-1}[y_U(m-1,p)], \tag{10b}$$
$$A(m,p)=(1-c)A(m-1,p), \tag{10c}$$

for $2^{m-2}+1\leqq p\leqq 2^{m-1}$

$$y_L(m,p)=f_2^{-1}[y_L(m-1,p-2^{m-2})], \tag{10d}$$
$$y_U(m,p)=f_2^{-1}[y_U(m-1,p-2^{m-2})], \tag{10e}$$
$$A(m,p)=cA(m-1,p-2^{m-2})+(1-c), \tag{10f}$$

with

$$y_L(1,1)=f_1^{-1}(y_U), \tag{11a}$$
$$y_U(1,1)=f_2^{-1}(y_L), \tag{11b}$$
$$A(1,1)=1-c, \tag{11c}$$

where y_L and y_U are defined as the solutions of the equations

$$y_L=f_1(y_L), \tag{12a}$$
$$y_U=f_2(y_U). \tag{12b}$$

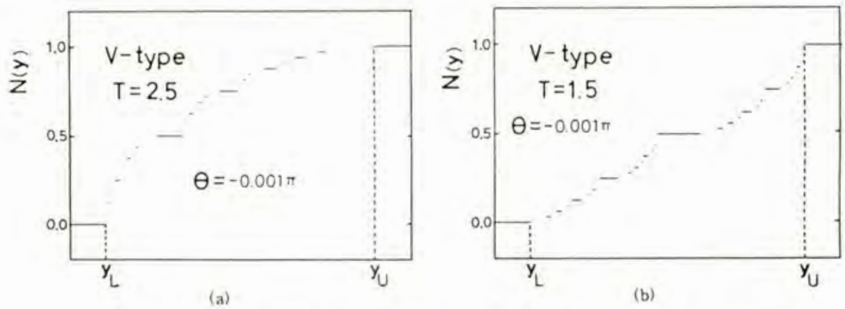

Fig. 1 The devil's staircase structure of the distribution function N(y) for V-type at (a) T=2.5 and (b) T=1.5. The 'stairs' upto fifth iteration level are shown. The bond distribution function is chosen as $P(J)=1/2$ $[\delta(J-1.0)+\delta(J-0.5)]$.

We note that the quantity y_L (y_U) corresponds to the value of y in the pure case, i.e. $y_{L,U} = \lim_{j\to\infty} y_j$ with $J^V(j) = J_1^V(J_2^V)$ for all j. Examples of the DS thus calculated from Eqs.(10)-(12) are shown in Fig. 1. The above arguments can be directly generalized to the H-type case where the functions $f_1(y)$ and $f_2(y)$ are the inverse functions of $F[y|J^V, J_1^H]$ and $F[y|J^V, J_2^H]$, respectively.

2.3 "Phase transition" of the distribution function

The DS structure is destroyed if the condition $y_L(1,1) \geqq y_U(1,1)$ is satisfied and there is a 'phase transition' in the shape of the distribution function between the DS-phase and the non-DS phase (Fig. 2). The phase diagram in T$-\theta$ plane is shown in Fig. 3, for both V-and H-type models. For high enough or low enough temperatures, we have the DS for all θ. On the other hand, the DS is always destroyed around $\theta=0$ in the intermediate range of temperatures including the interval $[T_1, T_2]$ where $T_1(T_2)$ is the critical temperature when all the vertical bond strength are set to be $J_1^V(J_2^V)$. We also find qualitatively identical behavior of the distribution function for H-type model.

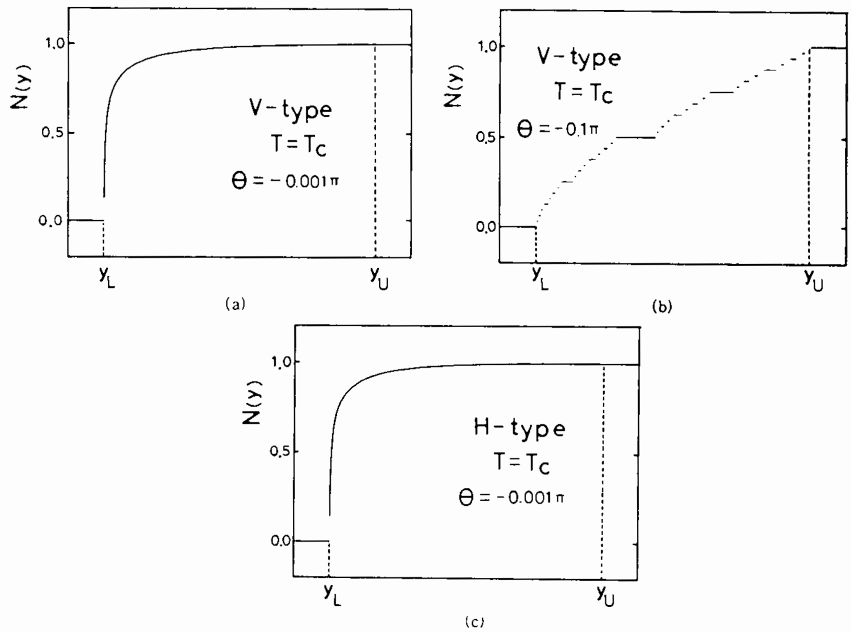

Fig. 2 The form of the distribution functiôn at $T=T_c$, (a) for small $|\Theta|$ where the devil's staircase is destroyed (V-type), (b) for large $|\Theta|$ where the devil's staircase is recovered (V-type). In (c), the similar figure as in (a) is shown for H-type case. The bond distribution function is chosen as $P(J)=1/2[\delta(J-1.0)+\delta(J-0.5)]$. The T_c (1.909 for V-type, 1.973 for H-type) is determined from Eq.(14). In the cases where the devil's staircase is destroyed, we have calculated N(y) from the distribution of y in the sequence $\{y_1,\cdots,y_M\}$ generated iteratively by Eqs.(4a,b).

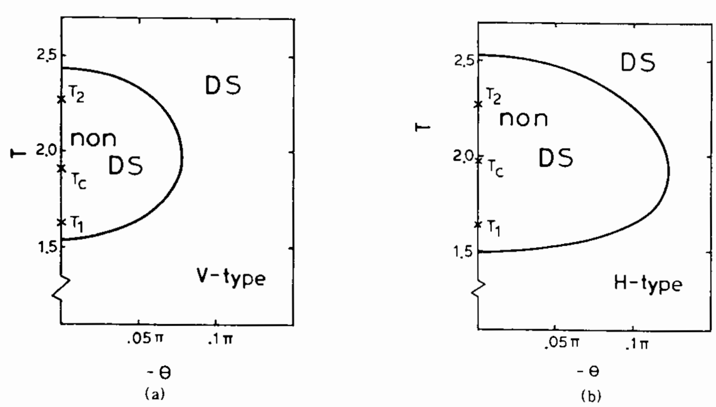

Fig. 3 The phase diagram of the distribution function for (a) V-type and (b) H-type. Two phases DS and non-DS are separated by the line determined from the condition $y_L(1,1)=y_U(1,1)$. We have chosen $P(J)=1/2[\delta(J-1.0)+\delta(J-0.5)]$ for both cases.

§3. Exact numerical method for calculation of thermodynamical quantities

3.1 Direct iteration method

We proceed to calculate thermodynamic quantities. In $M \to \infty$ limit, we can substitute the summations $(1/M)\sum_{j=0}^{M-1}$ in Eq.(2a,b) by integrals with suitable probability distributions as in the original treatment by McCoy-Wu. In particular, the 'singular' part of the free energy Fsing for V-type model can be expressed in a compact form as

$$F_{\text{sing}} = \left(-\frac{1}{\beta}\right)\frac{1}{4\pi}\int_{-\pi}^{\pi} d\theta \int dJ P(J) \int dy \nu(y) \ln\left[a^2 + b^2 + a\lambda(J)y\right] \tag{13}$$

It is, however, better to use the original formula (2b) and to make direct averages over the series $\{y_j\}$ generated iteratively, since there exist a range of temperatures where the DS is destroyed for some θ and the form of $\nu(y)$ $(=\partial N(y)/\partial y)$ is not known analytically. Moreover, by this 'direct' method, we can treat models with more general kinds of randomness, e.g. Gaussian type bond distribution, etc. We further note that, in the actual calculations, we should choose $\theta = (2n-1)\pi/N$ $(n=1,\cdots,N)$ in discretizing the θ integral in (2b) so that we may have a numerically exact solution for NxM system. Since the values of N and M can be chosen to be fairly large, we get a quite reliable answer to the original problem for the infinite system.

3.2 Behavior of the specific heat around the 'critical' temperature

The specific heat $C = -T(\partial^2 F/\partial T^2)$ is calculated in a similar way as mentioned in 3.1; We differentiate both sides of Eq.(2a) and use the differentiated version of the recursion equation (4a). Instead of writing the complicated expression, we present only the calculated results here. Note that due to the self-averaging character of the specific heat, sample (i.e. random number sequence) dependence is rather small for large N.

We focused on the behavior of the specific heat $C = -T\partial^2 F/\partial T^2$ around the 'critical' temperature determined by the equation[3,5]

$$\langle\langle K^H - K^{*V} \rangle\rangle = 0, \tag{14}$$

with $K^H = \beta J^H$ and K^{*V} being the dual vertical coupling $K^{*V} = -(1/2)\log[\tanh(\beta J^V)]$.

We first note that the T_c corresponds to the *peak position* of the specific heat

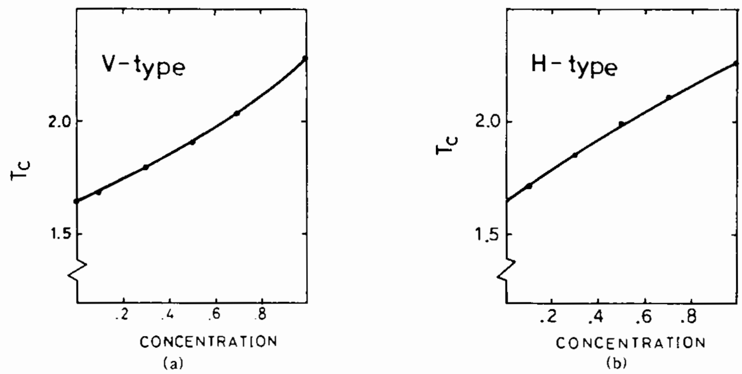

Fig. 4 Peak temperatures of the specific heat and the 'critical' temperature T_c, for (a) V-type and (b) H-type. Numerical results are represented by the dots. As a comparison, the 'critical' temperature T_c calculated from Eq. (14) is shown by the solid line. We have chosen $P(J)=c\,\delta\,(J-1.0)+(1-c)\,\delta\,(J-0.5)$ for both cases.

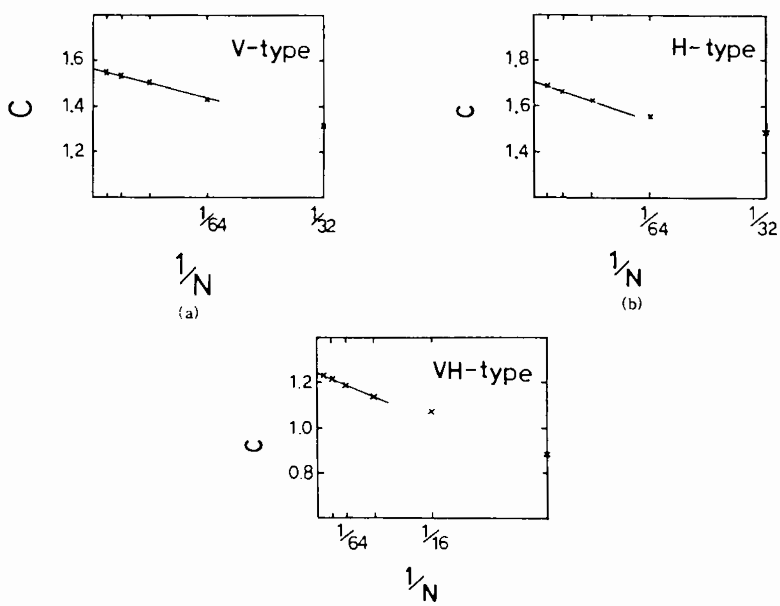

Fig. 5 Size dependence of peaked values of the specific heat for (a) V-type and (b) H-type. The calculated results of the specific heat are plotted versus 1/N. We have also shown the results for VH-type in (c). For all the cases we have chosen $P(J)=1/2[\,\delta\,(J-1.0)+\delta\,(J-0.5)\,]$.

(Fig. 4). This fact is also confirmed for H- and VH-type models. The size (N) dependence of the peak height is shown in Fig. 5. It seems that even in the thermodynamic limit $N \to \infty$, the peak height does not diverge.

§4. Summary and discussion

In this paper, McCoy-Wu type random Ising models are investigated for the random ferromagnetic case. For the case of binary bond distribution, behavior of the integrated distribution function N(y) is studied analytically and it is explicitly shown that N(y) forms the devil's staircase for some range of parameter values. There is a 'phase transition' of the distribution function and the phase diagram is drawn. Behavior of the specific heat is studied in numerically exact manner. The result shows that the critical temperature whose clear physical meaning has not been known corresponds to the peak position of the specific heat and that the peak height does not diverge even in the thermodynamic limit.

In the pure case, where $\nu(y)=\delta(y-y_L)$ (or $\nu(y)=\delta(y-y_U)$), the critical divergence of the specific heat comes from the singularity[3] (square-root branch point) of y_L (or y_U) at $\theta=0$ in the free energy around the pure critical temperature $T_2(T_1)$. In the random case, T_c lies in the interval $[T_1, T_2]$ and $T_c = T_1, T_2$. In this interval, the DS is destroyed near $\theta=0$, and then N(y) is smooth in y everywhere excluding its 'edges' at y_L and y_U (see Fig. 2). Hence the dominant contribution to the free energy comes from values of y around y_L or y_U. However, since $T_c=T_2$ (or T_1) hence y_L (y_U) is not singular, we do not have a singular behavior of the free energy.

In the light of the above argument, one may suppose that there should appear a singular behavior at T_1 or T_2. However, a detailed numerical calculation of the specific heat shows that there are no divergent behavior at these temperatures. At T_1 or T_2, we still have the non-DS behavior of the distribution function and the edge at y_L or y_U is sharp but somewhat smeared out. Hence the δ-function component at $y=y_L$ or $y=y_U$ of $\nu(y)$ is also smeared out. Therefore, the divergent contribution to the specific heat from $y=y_L$ or $y=y_U$ may be suppressed. Nevertheless, we conld not exclude nondivergent but singular behavior such as the essential singularity, which is known to be extremely difficult to observe.[9]

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Appendix

Though the exact solution of the generalized McCoy-Wu type random Ising model is obtained in §2, the thermodynamic quantities which are derived from this solution are restricted to the internal energy and the specific heat. In this appendix we obtain another important quantity, the boundary magnetization. As we can see in §2, a divergent behavior of the specific heat is drastically suppressed by the random bond distribution. Then, anomalous critical behavior of the spontaneous boundary magnetization is expected, i.e. a critical exponent would change.

To obtain the boundary magnetization we consider the M (vertical)xN(horizontal) system with periodic (free) boundary condition in the horizontal (vertical) direction. The magnetic field H is applied to the M-th row. Despite the applied field H this system is exactly soluvable by an excellent method of McCoy. For convenience we define variables $z=\tanh(\beta H)$, and $c=-2\sin\theta\,|1+e^{i\theta}|^{-2}$. The partition function of the system (MxN lattice with a boundary magnetic field) is equivalent to the one of a new lattice with an additional row which is connected to the original lattice by bonds of strength z. Then, obtaining the boundary magnetization is reduced to obtaining the free energy of the Ising model on a new lattice. This free energy is denoted by F(H). The boundary magnetization is derived from this free energy as

$$M=-\frac{\partial F(H)}{\partial H}=z+\frac{z(1-z^2)}{2\pi}\int_{-\pi}^{\pi}d\theta\frac{y_M}{c+z^2y_M}\,. \tag{A1}$$

Numerical analysis of Eq.(Al) is carried out by the direct iteration method used for the specific heat in §3. Since the boundary magnetization would not be a self averaging quantity, that is sample dependent even in the thermodynamic limit, we must take sample average over many sample realization. Typical behavior of averaged

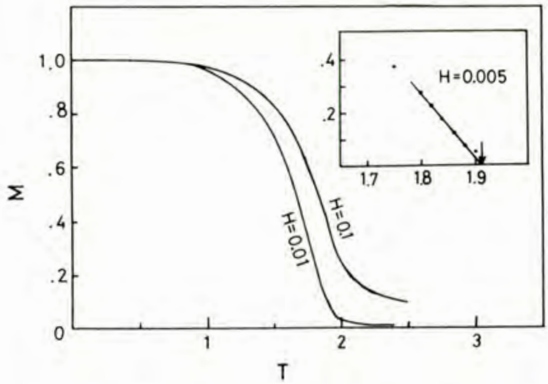

Fig. 6 The boundary magnetization versus temperature.

boundary magnetization is represented in Fig. 6. As shown in the inset of Fig. 6 , the boundary magnetization near 'the critical temperature' is well plotted by a straight line. Therefore, we can conculde that critical exponent β which is defined as

$$M \sim (T_c - T)^{\beta}, \tag{A2}$$

is 1 for the McCoy-Wu type random Ising model. For the pure system the spontaneous boundary magnetization vanishes at T_c as like $(T_c - T)^{1/2}$, i.e. $\beta = 1/2$. Therefore, we conclude that bond randomness affects the critical behavier of the system dramatically. It should be noted, here, that this result $\beta = 1$ is consistent with the one for the infinitely sharply peaked distribution[3]. We beleive that this conclusion holds for any type of randomness.

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